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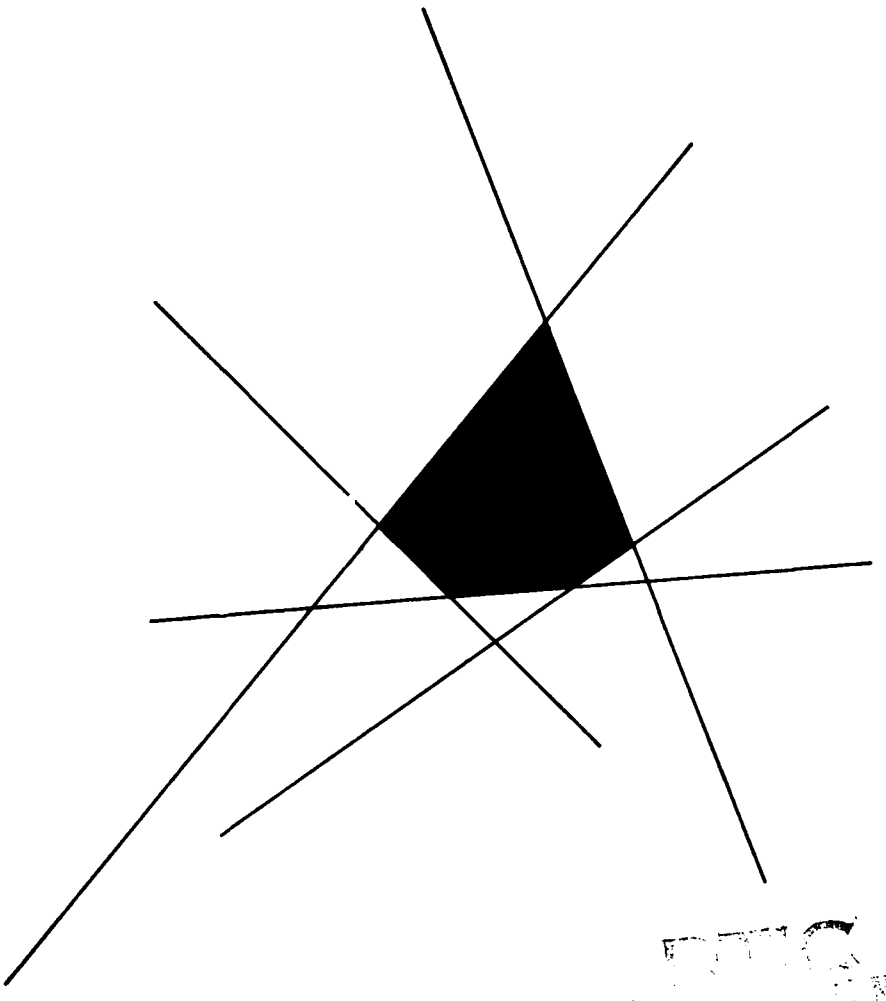
by
KING-TIM MAK



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GENERAL HOMOTHETIC PRODUCTION CORRESPONDENCES

by

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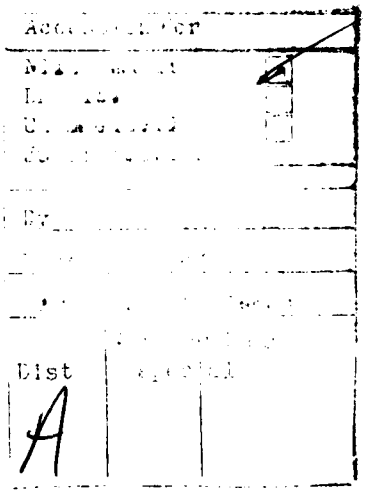

ABSTRACT

An abstract notion of the scaling of production factors is formulated. Through this formulation, homothetic production correspondences are generalized. Such generalization makes clear the basic structure of homotheticity and the associated expansion paths.

GENERAL HOMOTHETIC PRODUCTION CORRESPONDENCES

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1. INTRODUCTION

Production correspondences exhibiting certain scaling laws have been investigated over the years. Apart from the simple homogenous technologies, Shephard in [5], [6] introduced and studied homothetic and semi-homogenous structures and Eichhorn [2], [3] developed the class of quasi-homogenous production correspondences. It was shown in [4], that these various structures are special classes of the family of ray-homothetic structure, which in turn was characterized in terms of linear (proportional) expansion paths. Al-Ayat and Färe [1] then formulated the class of almost ray-homothetic production structure which includes ray-homotheticity while allowing for nonproportional changes in the inputs (outputs). Nonlinear expansion was also investigated there.

In this paper, scaling of production factors is formulated abstractly so as to encompass all the aforementioned structures. By doing this, insight into the structure of homotheticity is gained.

The arguments to follow are carried out within the framework of a production technology introduced in Shephard [6]. A mapping $x \to P(x) \in 2^{\mathbb{R}^m_+}$, of input vectors $x \in \mathbb{R}^n_+$ to subset $P(x)$ of all output vectors $u \in \mathbb{R}^m_+$ obtainable by x is called an output correspondence. Inversely, the input correspondence $u \to L(u) := \{x \mid u \in P(x)\}$ determines the set of all input vectors yielding the output vector $u \in \mathbb{R}^m_+$.

Both $L(u)$ and $P(x)$ are assumed to satisfy the inversely related set of weak axioms in [6]. Unless specifically indicated, free disposability of inputs or outputs is not enforced, nor is the convexity of $L(u)$ or $P(x)$.

2. SCALING OF PRODUCTION FACTORS AND GENERAL HOMOTHETICITY

General scaling operation on factor (input or output) space is modelled by

(2.1) Definition:

$T : \mathbb{R}^1_{++} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a *scaling operation* on space $\mathbb{R}^n_+$ if it satisfies:

(i) for all $(\mu,x) \in \mathbb{R}^1_{++} \times \mathbb{R}^n_+$, $T(\mu;\cdot) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is 1-1 and onto map, and $T(\cdot;x) : \mathbb{R}^1_+ \to \mathbb{R}^n_+$ is 1-1 map if $x \neq 0$;

(ii) $T(1;x) = x$; $T(\mu,0) = 0$ for all $\mu \in \mathbb{R}^1_{++}$;

(iii) $T(\mu;x) = y \iff T(1/\mu;y) = x$ for $\mu \in \mathbb{R}^1_{++}$;

(iv) $T(\lambda\cdot\mu;x) = T(\lambda;T(\mu;x))$ for all $(\lambda,\mu,x) \in \mathbb{R}^2_{++} \times \mathbb{R}^n_+$.

It should be noted that the above set of assumptions is not independent; in particular, (ii) and (iv) imply (iii): if $u > 0$ and $T(\mu;x) = y$, then $x = T(1;x) = T(1/\mu\cdot\mu;x) = T(1/\mu;T(\mu,x)) = T(1/\mu,y)$.

(2.2) Definition:

For given scaling operation T on $\mathbb{R}^n_+$, vector y is a *scaled version* of x , denoted yRx , if $\exists \lambda \in \mathbb{R}^1_{++}$ with $T(\lambda;x) = y$.

The relation R induced by T clearly satisfies

xRx , by (2.1-ii);

$yRx \iff xRy$, by (2.1-iii);

zRy and $yRx \Rightarrow zRx$, by (2.1-iv).

Thus, R generates equivalence classes of scaled versions of vectors. Denote the partition of space $\mathbb{R}^n_+$ via such equivalence classes by

$\mathcal{T} := \{C_\alpha\}_{\alpha\in A}$ where A is a collection of representative elements, one from each equivalence class. If $x' \in A$, then $C_{x'}$ is simply $\{x \mid xRx'\}$. The singleton $\{0\}$ belongs to $\mathcal{T}$. All this should be clear from the following example of usual (ray) scaling of vectors:

$$(\mu,x) \in \mathbb{R}^1_{++} \times \mathbb{R}^n_+ \to T(\mu;x) := \mu \cdot x \ ;$$

$A := \{y \mid ||y|| = 1\} \cup \{0\}$, $||\cdot||$ denotes Euclidean norm;

for $y \in A$, $C_y := \left\{\lambda \cdot y \mid \lambda \in \mathbb{R}^1_{++}\right\}$.

(2.3) Definition:

An output correspondence $x \to P(x)$ with scaling operation T on input space is called *scale homothetic* if it satisfies a functional equation of the form

$$P(T(\lambda;x)) = \psi(\lambda,x) \cdot P(x) \quad \text{for all} \quad (\lambda,x) \in \mathbb{R}^1_{++} \times \mathbb{R}^n_+ \ . \tag{2.4}$$

$$\psi : \mathbb{R}^1_{++} \times \mathbb{R}^n_+ \to \mathbb{R}^1_{++} \ , \ \psi(1,x) = \psi(\lambda,0) = 1 \quad \text{for all} \quad \lambda \in \mathbb{R}^1_{++} \ , \ x \in \mathbb{R}^n_+ \ .$$

If H is a scaling operation on output space, the scale homothetic input correspondence is defined analogously.

For simplicity, scaling operation will henceforth be denoted by the symbol $*$ (or $\circledast$), that is, with x, T (u,H) addressing to the input (output) space and $\lambda \in \mathbb{R}^1_{++}$

$$\lambda * u := T(\lambda;x) \ ; \ \lambda \circledast u := H(\lambda;u) \ .$$

(2.5) Proposition:

An output correspondence $x \to P(x)$ is scale homothetic if and only if $\exists F : \mathbb{R}^n_+ \to \mathbb{R}^1_+$ such that

$$P(\mu * x) = \frac{F(\mu * x)}{F(x)} \cdot P(x) \ , \ \mu \in \mathbb{R}^1_{++} \ . \tag{2.6}$$

Proof:

Since (2.4) implies (2.4), it is only necessary to prove (2.4) implies (2.6). For λ , $\mu \in \mathbb{R}^1_{++}$,

$$P((\lambda \cdot \mu) * x) = \psi(\lambda \cdot \mu, x) \cdot P(x) \quad \text{(by (2.4))},$$

also

$$\begin{aligned} P((\lambda \cdot \mu) * x) &= P(\lambda * (\mu * x)) \quad \text{by (2.1-iv)} \\ &= \psi(\lambda, \mu * x) P(\mu * x) = \psi(\lambda, \mu * x) \cdot \psi(\mu, x) \cdot P(x) \ . \end{aligned}$$

This implies ψ satisfies the functional equation

$$\psi(\lambda \cdot \mu, x) = \psi(\lambda, \mu * x) \cdot \psi(\mu, x) \ . \tag{2.7}$$

To solve this functional equation, an auxiliary function $f : \mathbb{R}^n_+ \setminus \{0\} \to \mathbb{R}^1_{++}$ is defined as follows:

(a) arbitrarily select a vector $\tilde{x}_\alpha$ from each of the equivalence class C_α , $C_\alpha \neq \{0\}$, where $\{C_\alpha\}_{\alpha \in A}$ is the partition by T ;

(b) for $y \in C_\alpha$, $f(y) := \mu$ iff $y = \mu * \tilde{x}_\alpha$.

Note that if $y \in C_\alpha$ and $y \neq 0$, we have $f(y) = \mu = T^{-1}(y; \tilde{x}_\alpha)$ where $T^{-1}(\cdot; \tilde{x}_\alpha)$ is the inverse to the function $T(\cdot; \tilde{x}_\alpha) : \mathbb{R}^1_+ \to \mathbb{R}^n_+$ with $\tilde{x}_\alpha$ fixed.

Suppose $y \in C_\alpha$, $y \neq 0$. Let $\mu := f(y)$. Since $\mu > 0$, $y \in C_\alpha$ implies $\mu * y \in C_\alpha$; and

$$f(\mu * y) = T^{-1}(\mu * y; \tilde{x}_\alpha)$$

$$= T^{-1}(\mu * T(\mu; \tilde{x}_\alpha); \tilde{x}_\alpha)$$

$$= T^{-1}(T(\mu^2; \tilde{x}_\alpha); \tilde{x}_\alpha)$$

$$= \mu^2 = \mu \cdot f(y) \ .$$

That is to say

$$f(\mu * y) = \mu \cdot f(y) \quad \text{for} \quad \forall (\mu, y) \in \mathbb{R}^1_{++} \times (\mathbb{R}^n_+ \setminus \{0\}) \ . \tag{2.8}$$

If $x = 0$, let $F(\lambda * x) \equiv 1$ in (2.6) for all $\lambda \in \mathbb{R}^1_{++}$, then (2.6) holds.

If $x \neq 0$, let $\lambda := 1/f(\mu * x)$, and rewrite (2.7) as

$$\psi(\mu / f(\mu * x), x) = \psi(1/f(\mu * x), \mu * x) \cdot \psi(\mu, x) \ .$$

From (2.8), it follows that

$$\psi(\mu, x) = \frac{\psi(1/f(x), x)}{\psi(1/f(\mu * x), \mu * x)} \ .$$

Defining

$$F(\mu * x) := [\psi(1/f(\mu * x), \mu * x)]^{-1} \tag{2.9}$$

and noting that $1 * x = x$, it follows that

$$\psi(\mu, x) = \frac{F(\mu * x)}{F(x)}$$

and (2.6) is established. Q.E.D.

(2.10) Examples:

(i) Ray-homothetic output correspondence:

$$(\lambda, x) \to T(\lambda; x) := \lambda \cdot x \ ;$$

$$P(T(\lambda; x)) = P(\lambda \cdot x) = \psi(\lambda, x) \cdot P(x) \ .$$

(ii) Almost ray-homothetic output correspondence:

Given $\alpha_i > 0$, $i = 1, \ldots, n$

$$(\lambda,x) \to T(\lambda;x) := \left(\lambda^{\alpha_1}\cdot x_1, \ldots, \lambda^{\alpha_n}\cdot x_n\right) = \lambda * x$$

$$P(\lambda * x) = \psi(\lambda,x) \cdot P(x) \ .$$

(iii) A scaling operation constructed by transformations:

Let functions $G_i : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ $(i = 1, \ldots, n)$ satisfy:

(a) $G_i(0) = 0$ and $G_i(\alpha) > 0$ if $\alpha > 0$;

(b) G_i nondecreasing and $G_i(\alpha) \to +\infty$ as $\alpha \to +\infty$;

(c) for x , $y \in \mathbb{R}^n_+$, $x \neq y$ implies $(G_1(x_1), \ldots, G_n(x_n)) \neq (G_1(y_1), \ldots, G_n(y_n))$.

Furthermore, let G_1 be invertible with inverse function $G_1^{-1}(\cdot)$. Define for $(\theta,x) \in \mathbb{R}^1_{++} \times \mathbb{R}^n_+$

$$T(\theta;x) := \begin{cases} 0 \ , \text{ if } \ x = 0 \ ; \\ \left(\dfrac{G_1(\theta\alpha)}{G_1(\alpha)} \cdot x_1, \ldots, \dfrac{G_n(\theta\alpha)}{G_n(\alpha)} \cdot x_n\right) \ \text{ where } \ \alpha = G_1^{-1}(x_1) \ , \text{ if } \ x_1 \neq 0 \ ; \\ \theta \cdot x \ , \text{ if } \ x \neq 0 \ , \ x_1 = 0 \ . \end{cases}$$

That $T(\cdot;\cdot)$ defines a scaling operation may be easily verified.

Note that if F is *scale-homogenous of degree* β $(\beta > 0)$, i.e.,

$$F(\mu * x) = \mu^{\beta} \cdot F(x) \ ,$$

then the output correspondence, as given by (2.6) is also *scale-homogenous degree* β , i.e.,

$$P(\mu * x) = \mu^{\beta} \cdot P(x) \ .$$

Another special case of interest is when $\psi(\mu,x)$ in (2.4) has the form $\psi(\mu,x) = \Delta(\mu,\phi(x))$ where ϕ is scale homogenous of degree β. Then (2.7) may be rewritten as

$$\Delta(\lambda\cdot\mu,\phi(x)) = \Delta(\lambda,\mu^{\beta}\cdot\phi(x)) \cdot \Delta(\mu,\phi(x)) \ , \tag{2.11}$$

which by manipulation (using (2.11) itself) gives

$$\Delta(\mu,\phi) = \frac{\Delta(\lambda\mu,\phi\cdot 1)}{\Delta(\lambda,\mu^{\beta}\phi\cdot 1)} = \frac{\Delta(\lambda\mu\cdot\phi^{1/\beta},1)/\Delta(\phi^{1/\beta},1)}{\Delta(\lambda\mu\cdot\phi^{1/\beta},1)/\Delta(\mu\cdot\phi^{1/\beta},1)}$$

where for simplicity, ϕ denotes $\phi(x)$. With the definition $\tilde{F}(\gamma) = \Delta(\gamma^{1/\beta},1)$, the solution to (2.11) is seen to be

$$\Delta(\mu,\phi(x)) = \frac{\tilde{F}(\mu^{\beta}\cdot\phi(x))}{\tilde{F}(\phi(x))} \ .$$

Then the output correspondence has the form similar to the usual ray-homothetic structure (see Färe and Shephard [4]):

$$P(\mu * x) = \frac{\tilde{F}(\mu^{\beta}\cdot\phi(x))}{\tilde{F}(\phi(x))} \cdot P(x)$$

with ϕ scale homogenous of degree β.

3. INVERSELY RELATED SCALE HOMOTHETIC STRUCTURES

In the last section, scale homothetic structure is defined for the output correspondence. In general, scale homothetic output structure does not imply the same for the input structure. However, if both the input and output correspondences are in some sense homothetic, special structures arise (as in the case that when both the input and output correspondences are ray-homothetic, they are semi-homogenous; see [3]). Two such special structures are investigated in this section.

(3.1) Definition:

Output correspondence $x \to P(x)$ with scaling operation T on input space is *semi-homogenous* if for each index α of the partition $\{C_\alpha\}_{\alpha \in A}$ induced by T, there exists positive scalar $g(\alpha)$ such that for $x \in C_\alpha$, $P(\lambda * x) = \lambda^{g(\alpha)} \cdot P(x)$.

(3.2) Definition:

Scaled disposability of input holds if $P(x) \subset P(\lambda * x)$ for all $\lambda \in [1, +\infty)$.

Similar definition is made for the input correspondence.

(3.3) Proposition:

Let the output correspondence P and the input correspondence L be ray-homothetic and scale-homothetic respectively; that is,

$$P(\lambda \cdot x) = \psi(\lambda, x) \cdot P(x) \quad \text{for all} \quad (\lambda, x) \in \mathbb{R}^1_{++} \times \mathbb{R}^n_{++},$$

$$\psi : \mathbb{R}^1_{++} \times \mathbb{R}^n_{+} \to \mathbb{R}^1_{++}, \ \psi(1, x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n_{+},$$

and

$$L(\mu \circledast u) = \chi(\mu, u) \cdot L(u) \quad \text{for all} \quad (\mu, u) \in \mathbb{R}^1_{++} \times \mathbb{R}^m_{++},$$

$$\chi : \mathbb{R}^1_{++} \times \mathbb{R}^m_{+} \to \mathbb{R}^1_{++}, \ \chi(1, u) = 1 \quad \text{for all} \quad u \in \mathbb{R}^m_{+}.$$

Moreover, for each $x \in \mathbb{R}^n_{++}$, $u \in \mathbb{R}^m_{++}$, let the functions $\psi(\cdot,x)$ and $\chi(\cdot,u)$ have inverses. Then the assumption of weak disposability of the inputs and the scaled outputs implies the semi-homogeneity of P and L in the usual sense (see [6] for definition of semi-homogenity).

Proof:

It is clear that the following relations are equivalent:

$$x \in L(\mu \circledast u) = \chi(\mu,u)L(u) \tag{3.4}$$

$$x/\chi(\mu,u) \in L(u) \tag{3.5}$$

$$u \in P(x/\chi(\mu,u)) = \psi(1/\chi(\mu,u),x)P(x) \tag{3.6}$$

$$x \in L\left(\frac{u}{\psi(1/\chi(\mu,u),x)}\right) . \tag{3.7}$$

Consequently (3.4) and (3.7) imply

$$L(u) = \frac{1}{\chi(\mu,u)} \cdot L\left(\frac{u}{\psi(1/\chi(\mu,u),x)}\right) . \tag{3.8}$$

With the assumption of weak disposal of inputs, for $\sigma > 1$, $x \in L(\mu \circledast u)$ implies $\sigma \cdot x \in L(\mu \circledast u)$. By repeating the argument (3.4) to (3.8) using $\sigma \cdot x$ instead of x, it follows that

$$\psi(1/\chi(\mu,u),\sigma \cdot x) = \psi(1/\chi(\mu,u),x) . \tag{3.9}$$

Now, using similar arguments as that leading to (2.7), the ray-homothecity of output correspondence P gives rise to the following functional equation:

$$\psi(\lambda\sigma,x) = \psi(\lambda,\sigma x) \cdot \psi(\sigma,x) . \tag{3.10}$$

In view of (3.9) and the assumption that $\chi(\cdot,u)$ has inverse, the solution to (3.10) is given by (see [3])

$$\psi(\lambda,x) = \lambda^{h(x/||x||)} \ , \ h(x/||x||) > 0 \ . \tag{3.11}$$

Then it follows from (3.8) and (3.11) that

$$L([\chi(\mu,u)]^{h(x/||x||)} \cdot u) = \chi(\mu,u) \cdot L(u) \ . \tag{3.12}$$

Note that if the ray $\{\theta y \mid \theta \geq 0\} \cap L(\mu \circledast u) \neq 0$ and $y \neq x$, $h(y/||y||)$ must equal to $h(x/||x||)$. Furthermore, the scale-homothecity of input correspondence $u \to L(u)$ implies that the ray $\{\theta x \mid \theta \geq 0\} \cap L(\mu \circledast u) \neq \emptyset$ if and only if $\{\theta x \mid \theta \geq 0\} \cap L(u) \neq \emptyset$. Thus, the exponent $h(x/||x||)$ in (3.12) really depends on the equivalent classes and has the form $h(\alpha)$ if u belongs to the αth equivalence class of the partition induced by the scaling operation H.

Finally, with $\chi(\cdot,u)$ invertible, we conclude from (3.12) that

$$L(\theta \cdot u) = \theta^{1/h(\alpha)} \cdot L(u) \ . \qquad \text{Q.E.D.}$$

Unfortunately, there does not appear to be any simple result if the input and output structures are both scale homothetic respectively. This is because the formula

$$P(\lambda * x) = \psi(\lambda,x) \cdot P(x)$$

implicitly applies the usual proportional (ray) scaling operation "$\cdot$" (see Example 2.10-i with $\alpha = 1$) on the output space which could be different from "$\circledast$." To resolve this difficulty, we redefine the notion of scale homotheticity. We again assume a scaling operation H on the output space, T on the input space as before, and denote them by $\circledast$ and $*$ respectively.

(3.13) Definition:

For all $\lambda \in \mathbb{R}^1_{++}$,

$$\lambda \circledast P(x) := \left\{ u \in \mathbb{R}^m_{++} \mid u = \lambda \circledast v \text{ for some } v \in P(x) \right\} ,$$

$$\lambda * L(u) := \left\{ x \in \mathbb{R}^n_{++} \mid x = \lambda * y \text{ for some } y \in L(u) \right\} .$$

(3.14) Definition:

An output correspondence $x \to P(x)$ has *general scale-homothetic structure* if it satisfies a functional equation

$$\begin{aligned} &P(\lambda * x) = \psi(\lambda, x) \circledast P(x) \ , \ \forall (\lambda, x) \in \mathbb{R}^1_{++} \times \mathbb{R}^n_{++} \ ; \\ &\psi : \mathbb{R}^1_{++} \times \mathbb{R}^n_{+} \to \mathbb{R}^1_{++} \ , \ \psi(1,x) = \psi(\lambda, 0) = 1 \ , \ \forall \lambda \in \mathbb{R}^1_{+} \ , \ x \in \mathbb{R}^n_{++} \ . \end{aligned} \tag{3.15}$$

(3.16) Definition:

An input structure $u \to L(u)$ has *general scale homothetic structure* if it satisfies a functional equation

$$\begin{aligned} &L(\mu \circledast m) = \chi(\mu, u) * L(u) \ , \ \forall (\mu, u) \in \mathbb{R}^1_{++} \times \mathbb{R}^m_{+} \ ; \\ &\chi : \mathbb{R}^1_{++} \times \mathbb{R}^m_{+} \to \mathbb{R}^1_{++} \ , \ \chi(1,u) = \chi(\mu, 0) = 1 \ , \ \forall \mu \in \mathbb{R}^1_{++} \ , \ u \in \mathbb{R}^m_{+} \ . \end{aligned} \tag{3.17}$$

Now, we may follow Eichhorn [3, Theorem 12.5.3] and establish the following

(3.18) Proposition:

Let both the output correspondence P and input correspondence L have general scale-homothetic structure, i.e., (3.15) and (3.17) hold. Moreover, let $\psi(\cdot, x)$ and $\chi(\cdot, u)$ both have inverses for each $x \in \mathbb{R}^n_{++}$

and $u \in \mathbb{R}^m_{++}$. Then the assumption of scale disposability (3.2) of both inputs and outputs implies the semi-homogeneity (3.1) of P and L .

Proof:

It is clear that the following relations are equivalent:

$$x \in L(\mu \circledast u) = \chi(\mu,u) * L(u) \tag{3.19}$$

$$1/\chi(\mu,x) * x \in L(u) \text{ , by (3.13) and (2.1-iii)} \tag{3.20}$$

$$u \in P(1/\chi(\mu,u)*x) = \psi(1/\chi(\mu,u),x) \circledast P(x) \tag{3.21}$$

$$\frac{1}{\chi(1/\chi(\mu,u),x)} \circledast u \in P(x) \tag{3.22}$$

$$x \in L\left(\frac{1}{\psi(1/\chi(\mu,u),x)} \circledast u\right) = \chi\left(\frac{1}{\psi(1/\chi(\mu,u),x)} ,u\right) * L(u) \text{ .} \tag{3.23}$$

Thus, by (3.19) and (3.23), for all $u \in S$, where $S = \left\{u \mid L(u) = \emptyset \text{ , } L(u) \neq \mathbb{R}^n_+\right\}$

$$\chi(\mu,u) = \chi\left(\frac{1}{\psi(1/\chi(\mu,u),x)} ,u\right) \text{ .} \tag{3.24}$$

Since $\chi(\cdot,u)$ has inverse, identity (3.24) implies

$$1/\mu = \psi\left(\frac{1}{\psi(\mu,u)} ,x\right) \text{ .} \tag{3.25}$$

Since $\psi(\cdot,x)$ has inverse $\psi^{-1}(\cdot,x)$, (3.25) may be written as

$$\psi^{-1}(1/\mu,x) \cdot \chi(\mu,u) = 1 \text{ .} \tag{3.26}$$

Using assumption of scale disposability of output, i.e.,

$$L(u) \subset L(\sigma \circledast u) \quad \text{for} \quad \sigma \in (0,1] \text{ ,}$$

we may repeat the argument from (3.19) on and start with $x \in L(\mu \circledast (\sigma \circledast u))$ instead of $x \in L(\mu \circledast u)$. Then we obtain, analogously to (3.26),

$$\psi^{-1}(1/\mu, x) \cdot \chi(\mu, \sigma \circledast u) = 1 \qquad (\sigma \in (0,1]) \ . \tag{3.27}$$

Equations (3.26) and (3.27) implies

$$\chi(\mu, \sigma \circledast u) = \chi(\mu, u) \ . \tag{3.28}$$

From (3.17) by taking $L(\mu\sigma \circledast u) = L(\mu \circledast (\sigma \circledast u))$, we obtain the following: for all $u \in S$

$$L(\mu\sigma \circledast u) = \chi(\mu\sigma, u) * L(u) \ , \tag{3.29}$$

also

$$\begin{aligned} L(\mu \circledast (\sigma \circledast u)) &= \chi(\mu, \sigma \circledast u) * L(\sigma \circledast u) \\ &= \chi(\mu, \sigma \circledast u) * (\chi(\sigma, u) * L(u)) = \chi(\mu, \sigma \circledast u) \cdot \chi(\sigma, u) * L(u) \ . \end{aligned} \tag{3.30}$$

Thus, for all $u \in S$, (3.28), (3.29) and (3.30) gives

$$\chi(\mu\sigma, u) = \chi(\mu, u) \cdot \chi(\sigma, u) \ . \tag{3.31}$$

And the solution of (3.31) yields

$$\chi(\mu, u) = \mu^{g(\beta)} \tag{3.32}$$

where $g : B \to \mathbb{R}^1_{++}$ and B is the index set for the partition $H := \{D_\beta\}_{\beta \in B}$ of output space induced by the scaling operation H (i.e., $\circledast$).

By similar argument applied to P , we obtain

$$\psi(\lambda, x) = \lambda^{h(\alpha)} \tag{3.33}$$

where $h : A \to \mathbb{R}^1_{++}$ and A is the index set for the partition $T := \{C_\alpha\}_{\alpha \in A}$ of input space induced by the scaling operation T (i.e., $*$).

Checking back, we see that χ and ψ as given by (3.32) and (3.33) satisfies

$$h(\alpha) \cdot g(\beta) = 1$$

for all pairs (α,β) for which exists $x \in C_\alpha$, $u \in D_\beta$ and $x \in L(u)$ with $u \in S$ and $x \in W := \{x \mid P(x) \neq \{0\}\}$.

In case $u \notin S$ or $x \notin W$, i.e., $L(u) = \emptyset$ or $L(u) = \mathbb{R}^n_+$ or $P(x) = \{0\}$, Equations (3.15) and (3.17) still apply v[illegible] 1-(iii)), and the proof is completed. Q.E.D.

4. EXPANSION PATHS - LINEAR AND NONLINEAR

A class of expansion paths will be considered in this section. For this purpose, define the cost minimization set $K(u,p)$ for input price $p \geq 0$, $u \geq 0$ with $L(u)$ not empty, by

$$K(u,p) = \{x \mid x \in L(u) \ , \langle p,x \rangle = Q(u,p)\} \ ,$$

where $Q(u,p)$ is the cost function given by

$$Q(u,p) = \min \{\langle p,x \rangle \mid x \in L(u)\} \ .$$

That $Q(u,p)$ is well defined and $K(u,p)$ not empty follows from the axioms of Shephard's technology.

(4.1) Definition:

Given $p \geq 0$ and $u \geq 0$ with $L(u)$ not empty, the expansion of output according to scaling operation H (denoted $\circledast$) has *linear input expansion path* if there exists a scalar valued function $(\theta,u) \to \chi(\theta,u)$ such that $K(\theta \circledast u,p) = \chi(\theta,u) \cdot K(u,p)$ for $\theta > 0$.

If the input structure $u \to L(u)$ is scale homothetic, the cost function satisfies

$$\begin{aligned} Q(\theta \circledast u,p) &= \min \{\langle p,x \rangle \mid x \in \chi(\theta,u) \cdot L(u)\} \\ &= \chi(\theta,u) \cdot Q(u,p) \ , \end{aligned}$$

for $\theta > 0$, $p \geq 0$ and $u \geq 0$ with $L(u)$ not empty. Hence, the cost minimization set is

$$K(\theta \circledast u,p) = \chi(\theta,u) \cdot K(u,p) \ . \qquad (4.2)$$

Thus, the expansion of output according to a scaling operation H for scale homothetic input structure has a linear input expansion path.

For the converse to hold, further conditions on the input structure L are imposed; namely, convexity and free disposal of inputs (i.e., $x' \geq x \in L(u) \Rightarrow x' \in L(u)$). The following lemma proved in [4] is of use.

(4.3) Lemma:

If $L(u)$ is convex for $u \in \mathbb{R}^m_+$ and inputs are freely disposable, then $L(u) = \bigcup_{p \geq 0} K(u,p) + \mathbb{R}^n_+$.

Now, assume the expansion according to scaling operation H has linear input expansion path, i.e., (4.2) holds for $\theta > 0$, $u \geq 0$ and $p \geq 0$. Since $\chi(\theta,u)$ is independent of p, it follows that

$$\bigcup_{p \geq 0} K(\theta \circledast u, p) = \chi(\theta,u) \cdot \bigcup_{p \geq 0} K(u,p) .$$

By adding $\mathbb{R}^n_+$ to both sides of the above expression and invoking Lemma (4.3), we see that $L(\theta \circledast u) = \chi(\theta,u) \cdot L(u)$.

Thus, we have established the following:

(4.4) Proposition:

If input structure L with scaling operation H (i.e., $\circledast$) is scale homothetic, then expansion of output according to H has a linear input expansion path. Furthermore, if the input sets $L(u)$ are convex and satisfy free disposability, the converse is also true.

The relationship of scale-homotheticity (on inputs) and linear output expansion paths may be established analogously.

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